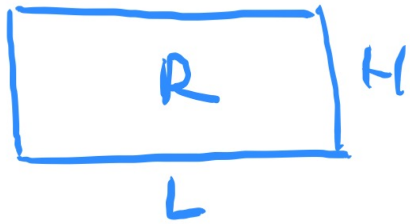
Recall: wave equation for rectangular region

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$



$$= c^2 \, \nabla^2 u$$

(same discussion more or less also applies $\underbrace{\text{to}}_{\text{2-dim}}$ heat equation

$$\frac{\partial u}{\partial t} = k \, \nabla^2 u$$

)

Consider product solution

$$u(x,y,t) = \phi(x,y)\, h(t)$$

$\rightarrow$ get eigenvalue problem $\boxed{\nabla^2 \phi = -\lambda \phi}$

<u>Result:</u> eigenvalues: $\lambda = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$

$n = 1, 2, \dots$
$m = 1, 2, \dots$

- eigenfunctions:

$$\sin\frac{n\pi}{L}x \, \sin\frac{m\pi}{H}y$$

<u>Fact:</u> Functions appearing in initial value conditions can be written as linear combinations of eigenfunctions

like $u(x,y,0) = \alpha(x,y)$

this was for boundary condition

$$u\big|_{\partial R} = 0$$

∂R = boundary of R

Similar pattern occur in greater generality.

- R can be any reasonable region
  (later: we will consider R = disk of radius a)

- more general boundary condition
  recall in 1-dim case:
  boundary condition could involve $\frac{\partial u}{\partial x}$ or u itself.

(Aside: $\exists$ general theory for 1-dim case
called Sturm-Liouville theory (Section 5)
where one considers general bd. condition

$$a_0 u(0,t) + b_0 \frac{\partial u}{\partial x}(0,t) = 0$$

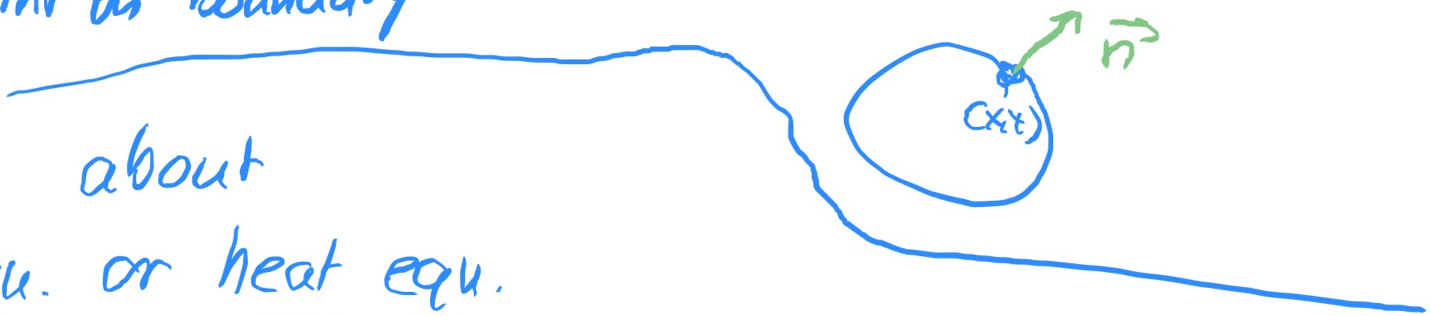
$$a_L u(L,t) + b_L \frac{\partial u}{\partial x}(L,t) = 0$$

Generalization for $\frac{\partial u}{\partial x}(0,t)=0 \to$ normal derivative $=0$

we consider general boundary conditions

$$a(x,y)\,\phi(x,y) + b(x,y)\,\underbrace{\nabla\phi(x,y)}_{\text{gradient}}\cdot\underset{\text{normal vector}}{\vec{n}} = 0 \qquad \text{BC}$$

(x,y) is point on boundary



General Facts about
2-dim wave equ. or heat equ.
with boundary cond. BC

(see list on p 283)

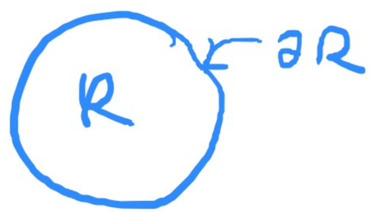
- $\exists$ product solution $u(x,y,t) = \phi(x,y)\,h(t)$
- $\nabla^2\phi = -\lambda\phi$ eigen function
- all eigenvalues λ are real
- infinitely many eigenvalues
- eigenspace for given λ may have dim > 1.
  (happens e.g. for rectangle if $H = L$)

- can expand functions in initial conditions as lin. comb. of eigenfunctions (generalization of Fourier series)

Wave/Heat Equation for a Disk R

wave equ.: $$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$ PDE

$$u\big|_{\partial R} = 0$$ BC



IC $$u(x,y,0) = \alpha(x,y)$$

$$\frac{\partial u}{\partial t}(x,y,0) = \beta(x,y)$$

Use polar coordinates:

Recall: in polar coordinates

$$\boxed{\nabla^2 \phi = \frac{1}{r} \partial/\partial r \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}}$$

where we write $u(x,y,t) = \phi(x,y)\, h(t)$

as for rectangle, we get eigenvalue problem

$$\boxed{\nabla^2 \phi = -\lambda \phi}$$

write $\phi(r,\theta) = f(r)\, g(\theta)$

$$\Rightarrow \frac{1}{r} \partial/\partial r \left(r\, \partial/\partial r \left(f(r)\, g(\theta) \right) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \left(f(r) g(\theta) \right) = -\lambda f(r) g(\theta)$$

$$\Rightarrow \frac{1}{r} \partial/\partial r \left(r\, \partial/\partial r\, f(r) \right) g(\theta) + \frac{1}{r^2} \frac{\partial^2 g}{\partial \theta^2} \cdot f(r) = -\lambda f(r) g(\theta) \quad \Big| \cdot \frac{r^2}{f \cdot g}$$

$$\Rightarrow \boxed{\frac{r\, \partial/\partial r \left(r f'(r) \right)}{f(r)} + \frac{g''(\theta)}{g(\theta)} = -\lambda r^2}$$

only depends on θ

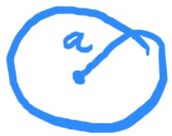
$= -\mu$

get $\boxed{g'' = -\mu g}$ i.e. $\frac{g''}{g} = -\mu$

and $\frac{r\,\partial/\partial r\,(r\,\partial/\partial r\, f)}{f} = -\lambda r^2 + \mu = -(\lambda r^2 - \mu)$

get $\boxed{r^2 f'' + r f' + (\lambda r^2 - \mu) f = 0}$

boundary conditions: $u(x,y,t) = 0$ on boundary of disk of radius a:



$\Rightarrow \phi(a,\theta) = 0$

$\Rightarrow f(a)\,g(\theta)$

$\Rightarrow \boxed{f(a) = 0}$

$-\pi$ and π denote same angle $\Rightarrow \boxed{\begin{aligned} g(-\pi) &= g(\pi) \\ g'(-\pi) &= g'(\pi) \end{aligned}} \rightarrow \boxed{\mu = n^2}$

$n = 1, 2, ..$

have seen for Laplace's equation for disk

that $\mu = n^2$, $n = 1, 2 \ldots$

plug into equ. for f

$$\Rightarrow \boxed{r \frac{d}{dr}\left(r \frac{df}{dr}\right) + \left(\lambda r^2 - m^2\right) f = 0}$$